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The Energy-Momentum Complex in Gravitational Induction

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Abstract

A restriction on the time-development of slowly changing axi-symmetric gravitational fields is discussed. The restriction, which is usually interpreted as the law governing near-field energy transfer, is obtained by a new approach using Komar's covariant energy-momentum superpotential. The particular form of the energy-conservation law is shown to be determined by the symmetries of the physical systems under consideration.

1. *Introduction*

The problem of near-field transfer of energy by gravitational fields has been investigated by Levy (1968), and by Morgan & Bondi (private communication), who have deduced a restriction for axial and reflection symmetric systems corresponding to the energy-flux theorem in Newtonian theory. Morgan and Bondi approached the problem by asking how a time sequence of static configurations, which can be deformed continuously into each other, differs from an arbitrary sequence of static configurations. The restriction which they found upon the sequence may be expressed in terms of the time dependence of 'multipole moments'

$$\frac{d}{dt}\left\{A_0 + \frac{1}{2}\sum_{l=0}^{\infty}(2l+1)A_l B_l\right\} = \frac{1}{2}\sum_{l=0}^{\infty}(A_l \dot{B}_l - \dot{A}_l B_l). \tag{1.1}$$

In Levy's approach, the field is taken as 'pseudo-Weyl', so that the metric is given by

$$g_{ij}(t,r,z) = a_{ij}(t,r,z) + h_{ij}(t,r,z),$$

where $a_{ij} = \mathrm{diag}[\exp(2u),\ -\exp(2k-2u),\ -\exp(2k-2u),\ -r^2\exp(-2u)]$, and h_{ij} is a first-order perturbation. In this case Levy shows that if Σ is a closed surface lying entirely in empty space, then two relevant field equations are

$$\begin{aligned} 0 = R_{01} &= S_1 + \Lambda_{,2}, \\ 0 = R_{02} &= S_2 - \frac{1}{r}(r\Lambda)_{,1}, \end{aligned} \tag{1.2}$$

$$\Lambda = \exp(2u-2k)\{\tfrac{1}{2}h_{02,1} - \tfrac{1}{2}h_{01,2} + h_{01}u_{,2} - h_{02}u_{,1}\}.$$

If these equations are to be integrable, then as a special case of De Rham's first theorem (Flanders, 1963),

$$\int_{\Sigma} \mathbf{S}.\mathbf{d\Sigma} = 0. \tag{1.3}$$

This condition is shown to be the same as the restriction (1.1), by evaluating the integral over a spherical surface Σ.

2. *The Energy-Momentum Superpotential*

There are of course an infinity of restrictions of the form (1.1) and (1.3) which may be taken, and it is to some extent a matter of expediency which of these is selected to represent an energy-conservation theorem. Some insight into the nature of the restriction proposed is gained by approaching the problem from the point of view of energy-momentum complexes. It is interesting that the condition (1.1) does in fact follow directly as an energy flux, by considering the Komar superpotential (Komar, 1959):

$$U^{[ij]} = \frac{\sqrt{-g}}{8\pi}(\xi^{i;j} - \xi^{j;i}).$$

This superpotential is particularly appropriate here, since although it is covariant in form, the physical systems that are considered single out a special symmetry from the infinity of general symmetries of nature. This means that if the quantities ξ^i are taken in the simple form (1,0,0,0) for the (t,r,z,ϕ) coordinate system, then an energy superpotential uniquely adapted to the physical situation will be obtained. It is shown here that this gives rise to an energy-flux law equivalent to the restriction (1.1) of Levy and of Morgan and Bondi.

The energy-flux law is derived from Komar's superpotential by first constructing strongly conserved quantities $\Phi^i = U^{[ij]}{}_{,j}$ which satisfy the relation

$$\Phi^i{}_{,i} = 0.$$

Weakly conserved quantities Ψ^i are then associated with Φ^i by writing

$$\Phi^i = \Psi^i - 2\sqrt{(-g)}\, G^i{}_j \xi^j = \Psi^i + \sqrt{(-g)}\, T^i{}_j \frac{\xi^j}{4\pi},$$

so that in empty space Ψ^i has vanishing divergence as a consequence of the field equations

$$\Psi^i{}_{,i} = 0.$$

3. *The 'Pseudo-Weyl' Metric*

In the case of the metric a_{ij}, where the coordinate conditions (Levy, 1968)

$$h_{11} = h_{22} = h_{33} = h_{44} = h_{12} = 0$$

are assumed, the Komar superpotential may be shown by means of extensive detailed calculation of the covariant derivatives $\xi^{i;j}$ to take a particularly simple form. All components of $U^{[ij]}$ vanish in the first-order approximation for the two components $U^{[12]}$ and $U^{[21]}$, which are given by

$$U^{[12]} = \frac{\Lambda r}{4\pi}, \qquad U^{[21]} = -\frac{\Lambda r}{4\pi}. \tag{3.1}$$

The non-vanishing components of Ψ^i are therefore given by

$$4\pi\Psi^1 = (r\Lambda)_{,2} - rR_{01},$$
$$4\pi\Psi^2 = -(r\Lambda)_{,1} - rR_{02}.$$

From equations (2.2) we therefore have on the surface Σ lying entirely in empty space,

$$-4\pi\Psi^1 = rS_1, \qquad -4\pi\Psi^2 = rS_2. \tag{3.2}$$

The vanishing divergence of Ψ^i therefore takes the form

$$\Psi^{\alpha}{}_{,\alpha} = 0, \qquad (\alpha = 1, 2, 3). \tag{3.3}$$

Integrating this relation over the three-dimensional region V, bounded by Σ,

$$\int_V \Psi^{\alpha}{}_{,\alpha}\, d^3 x$$

may be cast into a surface integral over the two-dimensional surface Σ by use of the generalised Stokes theorem:

$$\int_V f_{ij}\, d\tau^{ij} = \int_\Sigma f_{ij,k}\, d\tau^{ijk}.$$

Putting $f_{ij} = \Psi^{\alpha}\epsilon_{\alpha ij}/2!$, and calculating the cells $d\tau^{ij}$ and $d\tau^{ijk}$ in axial coordinates, the relation (3.3) gives

$$0 = \int_V \Psi^{\alpha}{}_{,\alpha}\, dr\, dz\, d\phi = \int_\Sigma \mathbf{S}.\mathbf{d\Sigma}.$$

The energy-flux law associated with the Komar superpotential is therefore expressed in the equation

$$\int_\Sigma \mathbf{S}.\mathbf{d\Sigma} = 0,$$

the same equation as is used by Levy, and which reduces on evaluation over a spherical surface to the restriction (2.1) on multipole moments. This approach to the analogue of the Newtonian Poynting vector shows clearly the relation of the restriction (2.1) to the theory of general conservation laws. The energy-flux theorem corresponding to (2.1) is in fact that one which is determined by the particular symmetry of the physical systems considered. The form of (2.1) has particular significance for these systems which would not necessarily be so for the general physical situation.

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